# ON THE FINITENESS OF STRESSES AT THE [LEADING] EDGE OF AN ARBITRARY CRACK

(O KONECHNOSTI NAPRIAZHENII NA KRAIU PROIZVOL'NOI TRESHCHINY)

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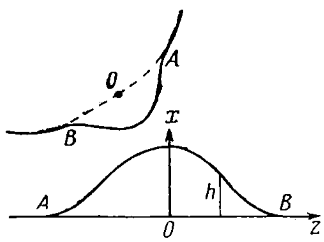
The condition of finiteness of the stresses and smoothness of closure at the [leading] edge of a crack of normal discontinuity has been suggested in hypothetical form by Khristianovich [1] in the nature of a basic condition defining the position of the ends of the crack. In [2] a proof of this condition was given for the case of cracks of normal discontinuity. In this paper we shall prove the finiteness of the stresses and the smoothness of closure of the crack at its boundary for an arbitrary equilibrium crack, i.e. for an arbitrary surface of discontinuity of displacements which is in equilibrium in an elastic body under the action of applied loads and cohesive forces.

**1.** Let us consider the region in the neighborhood of some point $O$ on the contour of the surface of an arbitrary discontinuity of displacements, i.e. a surface on which discontinuity occurs in all three components of the displacement vector. A system of orthogonal coordinates with center at the point $O$ is chosen such that the $xz$-plane is a contiguous plane to the contour of the surface of discontinuity at the point $O$, the $z$-axis is directed along the contour and the $x$-axis is directed into the body. It can be shown that the stress distribution on the $x$-axis near the origin is of the form

$$\sigma_y = \frac{N}{\sqrt{x}} + O(1), \quad \tau_{xy} = \frac{T_1}{\sqrt{x}} + O(1), \quad \tau_{yz} = \frac{T_2}{\sqrt{x}} + O(1), \quad \sigma_x,\ \tau_{xz},\ \sigma_z = O(1) \quad (1)$$

Here $\sigma_y$, $\tau_{xy}$, $\tau_{yz}$ are components of the stress tensor, $N$, $T_1$, $T_2$ are quantities which depend on the applied loads, the shape of the boundaries of the body and the contours of the cracks existing in the body and the position of the point $O$, but which are independent of $x$.

The displacements at points on the $x$-axis can be represented by the following formulas (see [3,4]):




$$u = -\frac{2(1+\nu)(1-2\nu)}{E} N\sqrt{x} + O(x^{3/2}) \qquad (x>0)$$

$$u = \pm\frac{2(1-\nu^2)}{E} T_1\sqrt{-x} + O(x^{3/2}) \qquad (x<0)$$

$$v = \frac{2(1+\nu)(1-2\nu)}{E} T_1\sqrt{x} + O(x^{3/2}) \qquad (x>0)$$

$$v = \pm\frac{2(1-\nu^2)}{E} N\sqrt{-x} + O(x^{3/2}) \qquad (x<0)$$

$$w = \pm\frac{2(1+\nu)}{E} T_2\sqrt{-x} \qquad (x<0)$$

$$w = 0 \qquad (x>0) \qquad (2)$$

Here $u$, $v$, $w$ are components of the displacement vector; $E$, $\nu$ are Young's modulus and Poisson's ratio, respectively, and the positive and negative signs correspond to the lower and upper edges of the crack, respectively.

**2.** From the actual state of equilibrium of the cracked elastic body we pass now to a possible state of equilibrium which differs from the former only in that the surface of discontinuity close to the point $O$ is slightly widened (see Figure), so that the new contour of the crack near the point $O$ is represented by a smooth curve, all points on which are situated close to $O$. The curve touches the previous contour at points $A$ and $B$. We shall evaluate the energy $\delta A$ released by this widening of the surface of discontinuity. We have an expression for $\delta A$ in the form of an integral over the new surface of the crack

$$\delta A = \frac{1}{2}2\int\{\sigma_y\delta v + \tau_{xy}\delta u + \tau_{yz}\delta w\}\,dS \qquad (3)$$

where

$$\delta u = \frac{2(1+\nu)}{E}\left[(1-2\nu)N\sqrt{x} + (1-\nu)T_1\sqrt{h-x}\right] + \cdots$$

$$\delta v = \frac{2(1+\nu)}{E}\left[-(1-2\nu)T_1\sqrt{x} + (1-\nu)N\sqrt{h-x}\right] + \cdots$$

$$\delta w = \frac{2(1+\nu)}{E} T_2\sqrt{h-x} + \ldots$$

(the dots denote small quantities of a higher order). The factor two appears, since the contributions to the energy released from the upper and lower edges of the crack are the same. The expression for the increment in energy is of the form

$$\delta A = \int_a^b dz\left\{\frac{2(1+\nu)}{E}\left[(1-\nu)(N^2+T_1^2) + T_2^2\right]\right\}\int_0^h\sqrt{\frac{h-x}{x}}\,dx =$$

$$= \frac{(1+\nu)\pi}{E} [(1-\nu)(N^2+T_1^2)+T_2^2] \int_a^b h\,dz$$

$$= \frac{(1+\nu)\pi}{E} [(1-\nu)(N^2+T_1^2)+T_2^2]\,\delta S \tag{4}$$

A corresponding formula for cracks of normal discontinuity is given by Irwin [5, 4].

But if the surface of discontinuity in question is an equilibrium crack, the energy $\delta A$ must vanish, from which and from (4) we obtain

$$(1-\nu)(N^2+T_1^2)+T_2^2 = 0 \tag{5}$$

Thus, evidently

$$N = T_1 = T_2 = 0$$

This proves the finiteness of the stresses and the smoothness of closure of opposite sides on the boundary of an arbitrary crack.

**3.** The foregoing discussion referred to a crack of arbitrary discontinuity on the surface of which the discontinuities in all three components of the displacement vector are nonzero, so that $[u] \neq 0$, $[v] \neq 0$, $w \neq [0]$; the symbol $[\ldots]$ denotes the difference in the values of the function on either side of the discontinuity. For three special types of cracks: cracks of normal discontinuity ($[u] = [w] = 0$, $[v] \neq 0$); transverse shear cracks ($[u] \neq 0$, $[v] = [w] = 0$); and longitudinal shear cracks ($[u] = [v] = 0$, $[w] \neq 0$) a more detailed examination is possible if we accept the hypothesis of smallness and discreteness of the end region of the surface of the crack [6]. The latter hypothesis amounts to the requirement that at all points on the boundary of the crack at which the intensity of cohesive forces is equal to the maximum possible one, the shape of a normal section of the end region and, consequently, the local distribution of cohesive forces, do not depend on the applied loading. It is evident that for cracks of arbitrary discontinuity the second hypothesis is unacceptable, and that for other types of cracks the shape of the end region of the surface of the crack is not the same.

Cracks of normal discontinuity have been investigated in detail already. For longitudinal and lateral shear cracks we obtain the boundary condition in an entirely analogous way [6] at points on the boundary at which the intensity of the cohesive forces is a maximum

$$T_1^0 = \frac{M}{\pi}, \qquad M = \int_0^d \frac{\tau_1(t)\,dt}{\sqrt{t}}, \qquad T_2^0 = \frac{L}{\pi}, \qquad L = \int_0^d \frac{\tau_2(t)\,dt}{\sqrt{t}} \tag{6}$$

where $T_1^\circ$ and $T_2^\circ$ are the corresponding quantities evaluated without taking into account cohesive forces, $d$ is the width of the end region of the crack, $\tau_1(t)$ and $\tau_2(t)$ are, respectively, the intensities of cohesive forces for cracks of both types. As in the case of a crack of normal discontinuity, the quantities $M$ and $L$ are constants of the material which define its resistance to the respective types of fracture.

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